Quantum mechanics II, Solutions 14 - Irreps of SO(3) and addition of angular momentum

TA : Achille Mauri, Behrang Tafreshi, Gabriel Pescia, Manuel Rudolph, Reyhaneh Aghaei Saem, Ricard Puig, Sacha Lerch, Samy Conus, Tyson Jones

Problem 1 : Clebsch-Gordan Coefficients

Here we consider breaking down tensor product representations of SO(3) into its irreducible representations.

— Use the ladder operators to show that $1 \otimes 1 = 2 \oplus 1 \oplus 0$.
Hint : The ladder operators for the composite system is

$$J_{\pm} = j_{\pm} \otimes 1 + 1 \otimes j_{\pm} \tag{1}$$

The first thing to remember is the ladder operator in the tensor product space.

$$J_{\pm} = j_{\pm} \otimes 1 + 1 \otimes j_{\pm} \tag{2}$$

Now, we use the highest weight decomposition method. The highest spin state is obviously $|J = 2, m = 2\rangle = |11\rangle$. Now, let us apply the lowering operator once :

$$J_{-} |11\rangle = \sqrt{2} |01\rangle + \sqrt{2} |10\rangle \tag{3}$$

Normalizing, we get $|J = 2, m = 1\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$. Applying the lowering operator once more :

$$J_{-}^2 |11\rangle = \sqrt{2} J_{-}(|10\rangle + |01\rangle) = \sqrt{2}(\sqrt{2} |00\rangle + \sqrt{2} |1, -1\rangle + \sqrt{2} |-1, 1\rangle + \sqrt{2} |00\rangle) \tag{4}$$

Normalizing, we get $|J = 2, m = 0\rangle = \frac{1}{\sqrt{6}}(|1, -1\rangle + 2 |00\rangle + |-1, 1\rangle)$. We proceed similarly :

$$\begin{aligned} J_{-}^3 |11\rangle &= 2J_{-}(|1, -1\rangle + 2 |00\rangle + |-1, 1\rangle) \\ &= 2\sqrt{2}(|0, -1\rangle + 2 |-1, 0\rangle + 2 |0, -1\rangle + |-1, 0\rangle) \end{aligned} \tag{5}$$

Normalizing, we get $|J = 2, m = -1\rangle = \frac{1}{\sqrt{2}}(|0, -1\rangle + |-1, 0\rangle)$. Finally,

$$\begin{aligned} J_{-}^4 |11\rangle &= 6\sqrt{2} J_{-}(|0, -1\rangle + |-1, 0\rangle) \\ &= 12(|-1, -1\rangle + |-1, -1\rangle) \end{aligned} \tag{6}$$

Normalizing, we get $|J = 2, m = -2\rangle = |-1, -1\rangle$. You can check that if you apply the lowering operator once more, you get 0, meaning that this is lowest state of this rep. To find the other sectors, we must first find the set of states which are orthogonal to the previous ones, and find the highest weight among those. Doing this, you get :

$$|J = 1, m = 1\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) \tag{7}$$

Remark : Note that this is just the vector orthogonal to $|J=2, m=1\rangle$ and built out of the same states. This is actually not a coincidence. Can you guess what happens in the general case ?

Now, again we apply the lowering operator :

$$J_-|J=1, m=1\rangle = \frac{1}{\sqrt{2}} J_-(|10\rangle - |01\rangle) = |00\rangle + |1,-1\rangle - |00\rangle - |-1,1\rangle = |1,-1\rangle - |-1,1\rangle \quad (8)$$

Therefore, $|J=1, m=0\rangle = \frac{1}{\sqrt{2}}(|1,-1\rangle - |-1,1\rangle)$ Applying the lowering operator once more :

$$J_-|J=1, m=0\rangle = |0,-1\rangle - |-1,0\rangle \quad (9)$$

Therefore, $|J=1, m=-1\rangle = \frac{1}{\sqrt{2}}(|0,-1\rangle - |-1,0\rangle)$. If we apply the lowering operator once more, we get zero which means that we have arrived at the lowest state. Finally, the only state orthogonal to all the above, is the unique state $|J=0, m=0\rangle = \frac{1}{3}(|1,-1\rangle - |00\rangle + |-1,1\rangle)$. You should check that $J_\pm |J=0, m=0\rangle = 0$, meaning that this is the 0. We can conclude that $1 \otimes 1 = 2 \oplus 1 \oplus 0$.

— Now show that $2 \otimes 1 = 3 \oplus 2 \oplus 1$

We follow similar to the previous part :

$$|J=3, m=3\rangle = |21\rangle \quad (10)$$

Apply lowering operator :

$$\begin{aligned} J_-|J=3, m=3\rangle &= 2|11\rangle + \sqrt{2}|20\rangle \\ \rightarrow |J=3, m=2\rangle &= \sqrt{\frac{2}{3}}|11\rangle + \frac{1}{\sqrt{3}}|20\rangle \end{aligned} \quad (11)$$

Again :

$$\begin{aligned} J_-|J=3, m=2\rangle &= 2|01\rangle + \frac{2\sqrt{3}}{3}|10\rangle + \frac{2\sqrt{3}}{3}|10\rangle + \sqrt{\frac{2}{3}}|2,-1\rangle \\ \rightarrow |J=3, m=1\rangle &= \sqrt{\frac{1}{15}}|2,-1\rangle + \sqrt{\frac{8}{15}}|10\rangle + \sqrt{\frac{2}{5}}|01\rangle \end{aligned} \quad (12)$$

Again :

$$\begin{aligned} J_-|J=3, m=1\rangle &= 2\sqrt{\frac{1}{15}}|1,-1\rangle + 4\sqrt{\frac{1}{5}}|00\rangle + 4\sqrt{\frac{1}{15}}|1-1\rangle + 2\sqrt{\frac{3}{5}}|-1,1\rangle + 2\sqrt{\frac{1}{5}}|00\rangle \\ \rightarrow |J=3, m=0\rangle &= \frac{1}{\sqrt{5}}|1,-1\rangle + \sqrt{\frac{3}{5}}|00\rangle + \frac{1}{\sqrt{5}}|-1,1\rangle \end{aligned} \quad (13)$$

At this point you should know how to proceed. We provide the answers so that you can check the final results :

$$\begin{aligned} |J=3, m=-1\rangle &= \sqrt{\frac{1}{15}}|-2,1\rangle + \sqrt{\frac{8}{15}}|-10\rangle + \sqrt{\frac{2}{5}}|0,-1\rangle \\ |J=3, m=-2\rangle &= \sqrt{\frac{2}{3}}|-1,-1\rangle + \frac{1}{\sqrt{3}}|-2,0\rangle \\ |J=3, m=-3\rangle &= |-2,-1\rangle \end{aligned} \quad (14)$$

Again, if you apply the lowering operator once more, you get zero. Now, look at the complementary subspace and find the highest weight (the state which is annihilated by the raising operator).

Remark : The trick is again to just look at $|J=3, m=2\rangle$ and find the orthogonal vector to it consisting only the same vectors !

That is :

$$\begin{aligned}
|J=2, m=2\rangle &= \sqrt{\frac{2}{3}}\,|20\rangle - \frac{1}{\sqrt{3}}\,|1,1\rangle \\
|J=2, m=1\rangle &= \frac{1}{\sqrt{3}}\,|2,-1\rangle + \frac{1}{\sqrt{6}}\,|1,0\rangle - \frac{1}{\sqrt{2}}\,|0,1\rangle \\
|J=2, m=0\rangle &= \frac{1}{\sqrt{2}}\,|1,-1\rangle - \frac{1}{\sqrt{2}}\,|-1,1\rangle \\
|J=2, m=-1\rangle &= -\frac{1}{\sqrt{3}}\,|-2,1\rangle - \frac{1}{\sqrt{6}}\,|-1,0\rangle + \frac{1}{\sqrt{2}}\,|0,-1\rangle \\
|J=2, m=-2\rangle &= -\sqrt{\frac{2}{3}}\,|-2,0\rangle + \frac{1}{\sqrt{3}}\,|-1,-1\rangle
\end{aligned} \tag{15}$$

Now, for the final time, we fine the complementary subspace to all the above vectors, find the highest weight and do the procedure. The final answer is :

$$\begin{aligned}
|J=1, m=1\rangle &= \sqrt{\frac{3}{5}}\,|2,-1\rangle - \sqrt{\frac{3}{10}}\,|10\rangle + \frac{1}{\sqrt{10}}\,|01\rangle \\
|J=1, m=0\rangle &= \sqrt{\frac{3}{10}}\,|1,-1\rangle - \sqrt{\frac{2}{5}}\,|00\rangle + \sqrt{\frac{3}{10}}\,|-1,1\rangle \\
|J=1, m=-1\rangle &= -\sqrt{\frac{3}{5}}\,|-2,1\rangle + \sqrt{\frac{3}{10}}\,|-1,0\rangle - \frac{1}{\sqrt{10}}\,|0,-1\rangle
\end{aligned} \tag{16}$$

— What does this tell you about the addition of angular momentum ?

— An application of the Clebsch-Gordon coefficients is the Wigner-Eckart theorem that you have seen during the lectures. Let us see an easy example of how it works. First we should understand what it means that an operator transforms under $SO(3)$. Take the position operator, x. We can write it in terms of spherical harmonics. Given that :

$$\begin{aligned}
Y_1^{-1} &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x-iy}{r} \\
Y_1^{0} &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{z}{r} \\
Y_1^{1} &= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x+iy}{r}
\end{aligned} \tag{17}$$

Can you write the position operator in terms of these spherical harmonics ?

Let us define (for convenience) :

$$T_q^1 = \sqrt{\frac{4\pi}{3}} r Y_1^q. \tag{18}$$

Then the position operator can be written as :

$$x = \frac{T_{-1}^1 - T_1^1}{\sqrt{2}} \tag{19}$$

— The spherical harmonics Y_l^m form a representation of SO(3). This is just similar to what you saw in the previous problem set(part 2 of problem 1, set 13). When we say that an operator transforms under $SO(3)$, this means that $Y_l^m |J, M\rangle$ transforms the same as $|l, m\rangle |J, M\rangle$. How does x, transform under $SO(3)$?

x transforms like $\frac{1}{\sqrt{2}}(|J = 1, M = -1\rangle - |J = 1, M = 1\rangle)$, where the first term corresponds to T_1^{-1} and the second term corresponds to T_1^1.

— Assume that we want to calculate $\langle n, j, m| \hat{x} |n, j, m\rangle$. Using the Wigner-Eckart theorem, what can you say without calculating any integrals ?

By Wigner-Eckart, we have :

$$\begin{aligned} \langle n, j, m| x |n, j, m\rangle &= \langle n, j, m| \frac{T^1_{-1} - T^1_1}{\sqrt{2}} |n, j, m\rangle \\ &= \frac{1}{\sqrt{2}} \langle n, j| |T^1| |n, j\rangle (\langle j, m|j, m, 1, -1\rangle - \langle j, m|j, m, 1, 1\rangle) \end{aligned} \tag{20}$$

Notice that both terms in the parenthesis are zero by Clebsch-Gordon. This means that this amplitude is zero by group theory! Moreover, the power of Wigner-Eckart is through the fact that every operator can be expanded in terms if harmonic functions (because harmonic functions form an orthonormal basis of the space of functions) and then we can apply the theorem. This means that there is nothing special about the x operator. Can you do the same argument for y ?

Problem 2 : Two interacting spins

Consider two spins $\hat{\mathbf{S}}_1$, $\hat{\mathbf{S}}_2$, with spin 1 ($\hat{\mathbf{S}}_1^2 = \hat{\mathbf{S}}_2^2 = S(S+1) = 2$) described by the Hamiltonian

$$H = J(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) - b_1 \hat{S}_1^z - b_2 \hat{S}_2^z \ . \tag{21}$$

1. Consider first the case in which $b_1 = b_2 = 0$. What is the symmetry group of the Hamiltonian in this case ? What are the corresponding conserved quantities (think in terms of commutator with the Hamiltonian) ?

   When $b = 0$, the system is invariant under spin rotations and under permutations of the two spins. The spin rotations are described by the SO(3) group (the extension to SU(2) is here unnecessary since the spin is integer and not half-integer). The SO(3) group is continuous and has as infinitesimal generator the total angular momentum $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$. Thus, $\hat{\mathbf{J}}$ is a conserved quantity.

   It can be checked in fact that

$$[\hat{S}_1^\alpha + \hat{S}_2^\alpha, J(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)] = iJ\epsilon^{\alpha\beta\gamma}(\hat{S}_1^\gamma \hat{S}_2^\beta + \hat{S}_2^\gamma \hat{S}_1^\beta) = 0 \ . \tag{22}$$

   This equation, as usual, has a double interpretation. It can be viewed as the statement that the Hamiltonian is invariant under rotations $[\hat{S}, \hat{H}] = 0$. On the other hand, it also means that the total spin is invariant under time translations, since the Hamiltonian is the generator of time translations. Thus the total spin is conserved.

   The symmetry under permutations is discrete and is described by the Z_2 group. Thus the full invariance group is $Z_2 \times$SO(3). The permutation operation commutes with all rotations.

2. Calculate the energy spectrum and the degeneracies of the energy levels for $b_1 = b_2 = 0$. *Hint.* Express the scalar product $\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$ in terms of the total angular momentum.

   Since the Hamiltonian has SO(3) symmetry, we can diagonalize it simultaneously with $\hat{\mathbf{S}}^2$ and with $\hat{S}^z$. This implies that energy levels can be labeled with the quantum numbers S, S^z, corresponding to the total angular momentum.

   By the addition of angular momenta, $1 \otimes 1 = 0 \oplus 1 \oplus 2$ we can know that the quantum number S can take three values, $S = 0, 1, 2$. The three values $S = 0$, $S = 1$, $S = 2$ correspond thus to three energy levels of the system. Due to the SO(3) symmetry, the level with angular momentum S must have degeneracy $2S + 1$, and the energy cannot depend on the quantum number S^z whose values range from $-S$ to $+S$.

   The energy can be calculated explicitly as follows. The scalar product $(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)$ can be expressed by completing the square as

$$(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) = \frac{1}{2}[(\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2 - \hat{\mathbf{S}}_1^2 - \hat{\mathbf{S}}_2^2] = \frac{1}{2}[S(S+1) - 2 \times 1(1+1)] = \frac{1}{2}[S(S+1) - 4] \ . \tag{23}$$

   Thus the energy levels are $E(S) = J[S(S+1) - 4]/2$. In particular, note that there is no "accidental degeneracy" between levels with different values of S.

   For $J > 0$ the ground state is a "singlet" with zero angular momentum and energy $E = -2J$. The first excited state is a "triplet" with $S = 1$, and $E = -J$. The second excited state, fivefold degenerate, has $S = 2$ and $E = +J$.

3. Consider now the case in which $b_1 \neq b_2 \neq 0$. What is the symmetry group, and what are the associated conserved quantities ? Without explicitly solving the problem, discuss how many energy levels you expect, and what are the corresponding degeneracies.

When b_1 and b_2 are different from zero, the SO(3) symmetry is broken, as there is a special direction in space (in this case the z direction), which is selected by the external fields b_1, b_2. The system however retains an SO(2) symmetry, under rotations with an axis along z. Thus, $\hat{S}^z$ remains conserved and S^z is still a good quantum number. For $b_1 \neq b_2$ the Z_2 group (permutation $\mathbf{S}_1 \leftrightarrow \mathbf{S}_2$ of the two spins) is not a symmetry of the Hamiltonian. Thus, the symmetry is reduced to SO(2).

From group theoretical arguments, we can deduce that S^z is a good quantum number. This simplifies the problem significantly, as we can reduce the initial problem (the diagonalization of a 9×9 matrix) into a block diagonalization, where the blocks correspond to the possible values of S^z. However, we can derive less conclusions in comparison with the previous case, where SO(3) symmetry was present.

For example, consider the states with $S^z = 0$. There are three of these states, $|+1,-1\rangle$, $|0,0\rangle$, $|-1,+1\rangle$. In presence of SO(3) symmetry, the eigenstates of the Hamiltonian must be eigenstates not only of $\hat{S}^z$, but also of $\hat{\mathbf{S}}^2$. This fixes-using only symmetry arguments-the precise superpositions of $|+1,-1\rangle$, $|0,0\rangle$, $|-1,+1\rangle$ which constitute the eigenstates of H. When $b_1 \neq b_2 \neq 0$, instead, the eigenstates belonging to the sector $S^z = 0$ are superpositions of $|+1,-1\rangle$, $|0,0\rangle$, $|-1,+1\rangle$ which have to be calculated by explicitly diagonalizing the Hamiltonian inside the block.

In the case $b_1 \neq b_2$, group theory does not imply any degeneracy at all in the spectrum. For generic values of the parameters, the spectrum will consist of 9 nondegenerate levels (9=3×3 is the total dimension of the Hilbert space).

In group-theory language, SO(2) is an Abelian group and thus has only one-dimensional irreducible representations. Thus, a system which has only SO(2) symmetry and no other symmetry, does not have a syemmtry-related degeneracy (although it may have accidental degeneracies).

4. Calculate the spectrum explicitly in the case in which $b_1 = b_2 = b \neq 0$.

   For $b_1 = b_2$, the symmetry becomes $Z_2\times$ SO(2). The group is still Abelian, because the rotations and the permutation commute among each other. Thus we still expect a spectrum consisting of 9 nondegenerate levels.

   The problem can be solved by a trick. The Hamiltonian for $b_1 = b_2$, in fact, has the property that it commutes with $\hat{\mathbf{S}}^2$. This is not a consequence of any symmetry, but just a simple property of the specific Hamiltonian considered here. Then, S remains a good quantum number, and the energies are $E = J[S(S+1) - 4]/2 - bS^z$.

   The fact that S is a good quantum number is not a consequence of symmetry : the symmetry here is broken to SO(2) and thus there is no group theoretical reason why S should remain a good quantum number.

   To proceed more systematically, using only the real group theoretical information which we have we can explicitly block-diagonalize the Hamiltonian. The blocks are given by irreducible representations of the group $Z_2\times$ SO(2) group. The sectors with $S^z = 2$ have only one state, so the block is one-dimensional and the block matrix is 1×1. This means that $|+1,+1\rangle$ and $|-1,-1\rangle$ are exact eigenstates. Their energy is

$$E(S^z = +2) = \langle +1,+1|H|+1,+1\rangle = J - 2b \; . \tag{24}$$

$$E(S^z = -2) = \langle -1,-1|H|-1,-1\rangle = J + 2b \; . \tag{25}$$

   For the $S^z = 1$ sector we have two states $|+1,0\rangle$, $|0,+1\rangle$. Using Z_2 symmetry we can further reduce the blocks considering symmetric and antisymmetric combinations. We arrive again at two exact states, $(|+1,0\rangle + |0,+1\rangle)/\sqrt{2}$, $(|+1,0\rangle - |0,+1\rangle)/\sqrt{2}$ which must be exact eigenstates.

Their energies are

$$E(S^z = +1, p = +1) = J - b \tag{26}$$

$$E(S^z = +1, p = -1) = -J - b \tag{27}$$

In the expression, p is the "parity" quantum number associated with the Z_2 group. $p = 1$ for states which are even under permutation of the two spins and $p = -1$ for states that are odd. Similarly for $E(S^z = -1, p)$ we find $E(S^z = -1, p = 1) = J + b$, $E(S^z = -1, p = -1) = -J + b$. Thus the states are degenerate with those having $E(S^z = +1, p)$. This degeneracy is accidental and not related to symmetry because the corresponding states are in different irreducible representations of $Z_2 \times SO(2)$.

For the $S = 0$ sector, we have three states : $|+1,-1\rangle$, $|0,0\rangle$, and $|-1,+1\rangle$. Due to the Z_2 symmetry, the states which are respectively even and odd under permutation of the two spins cannot mix. Thus the state $(|+1,-1\rangle - |-1,+1\rangle)/\sqrt{2}$, which is odd, must be an exact eigenstate. The corresponding energy can be calculated explicitly.

$$\begin{aligned} E(S^z = 0, p = -1) &= \frac{1}{2}\left(\langle +1,-1| - \langle -1,+1|\right) H \left(|+1,-1\rangle - |+1,-1\rangle\right) \\ &= \frac{1}{2}\left(\langle +1,-1| - \langle -1,+1|\right)\left(J\hat{S}_1^z\hat{S}_j^z - b_1\hat{S}_1^z - b_2\hat{S}_2^z\right)\left(|+1,-1\rangle - |+1,-1\rangle\right) \end{aligned} \tag{28}$$

The calculation simplifies because the operators S_i^x, S_i^y have non-zero matrix elements only between states for which the eigenvalue of S^z changes by ± 1. The result is :

$$E(S^z = 0, p = -1) = -J\ . \tag{29}$$

Finally, we have to consider the two states $|0,0\rangle$, and $\frac{1}{\sqrt{2}}(|+1,-1\rangle + |-1,+1\rangle)$. For these we have to block-diagonalize a 2×2 Hamiltonian

$$h_{S=0,p=+1} = \begin{vmatrix} \langle 0,0|H|0,0\rangle & \langle 0,0|H|\phi\rangle \\ \langle \phi|H|0,0\rangle & \langle \phi|H|\phi\rangle\ , \end{vmatrix} \tag{30}$$

where $|\phi\rangle = (|1,-1\rangle + |-1,1\rangle)/\sqrt{2}$. Calculating the matrix explicitly gives

$$h_{S=0,p=+1} = \begin{vmatrix} 0 & \sqrt{2}J \\ \sqrt{2}J & -J \end{vmatrix} \tag{31}$$

The eigenvalues are $J/2 \pm 3J/2 = 2J, -J$. Thus, as found also above, we see that the spectrum has the nine levels $2J + 2b$, $2J+$

Group theory in a nutshell version :

Let's now consider the case of a particle moving in 1D in a periodic potential $V(x)$. That is under the Hamiltonian

$$H = \frac{p^2}{2m} + V(x) \quad \text{where} \quad V(x+a) = V(x)\,. \tag{32}$$

We will suppose that the particle moves on a 1-dimensional lattice consisting of N sites and periodic boundary conditions.

What is the symmetry in group in this case? Well the Hamiltonian is left unchanged by any translation U_a by a distance a, i.e., $x \to Ux = x + a$. It follows, that the symmetry group consists of $\{I, U_a, U_a^2, \cdots, U_a^{N-1}\}$. Note that given the periodic boundary conditions we have that $U_a^N = I$. Thus the symmetry group is just the familiar cyclic group $\mathbb{Z}_N$. In the problem sheet, you'll then use your understanding of the irreps of $\mathbb{Z}_N$ to determine the form of the eigenfunctions of H.

Since the group is abelian, it can only have 1-dimensional irreducible representations. Hence $\psi'(x) \equiv T\psi(x) = \zeta\psi(x)$, so that T is represented by a complex number ζ. But since wave functions are normalized, $\int dx|\psi'(x)|^2 = \int dx|\psi(x)|^2$, we have $|\zeta| = 1$. Thus, we can write $\zeta = e^{ika}$ with k real (and with dimension of an inverse length). Since $e^{i(ka+2\pi)} = e^{ika}$, we may restrict k to the range

$$-\frac{\pi}{a} \leq k \leq \frac{\pi}{a}$$

This range is the simplest example of a Brillouin zone.

The symmetry group is just the familiar cyclic group $\mathbb{Z}_N$. The condition $e^{iNka} = 1$ thus implies that $k = \frac{2\pi}{Na}j$ with j an integer. For N macroscopically large, the separation Δk between neighboring values of j, of order $\frac{2\pi}{Na}$, is infinitesimal, and so we might as well treat k as a continuous variable ranging from $-\frac{\pi}{a}$ to $\frac{\pi}{a}$.

It is convenient and conventional to write

$$\psi(x) = e^{ikx}u(x) \tag{33}$$

with $u(x + a) = u(x)$. This statement is known as Bloch's theorem. (Of course, any $\psi(x)$ can be written in the form (11); the real content is the condition $u(x + a) = u(x)$.)

Note that this is a general statement completely independent of the detailed form of $V(x)$. Given a specific $V(x)$, the procedure would be to plug (11) into the Schrödinger equation and solve for $u(x)$ for the allowed energy eigenvalues, which of course would depend on k and thus can be written as $E_n(k)$. As k ranges over its allowed range, $E_n(k)$ would vary, sweeping out various energy bands labelled by the index n.